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Weighted L^p-Hardy and L^p-Rellich inequalities with boundary terms on stratified Lie groups

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Abstract
In this paper, generalised weighted L^p-Hardy, L^p-Caffarelli–Kohn–Nirenberg, and L^p-Rellich inequalities with boundary terms are obtained on stratified Lie groups. As consequences, most of the Hardy type inequalities and Heisenberg–Pauli–Weyl type uncertainty principles on stratified groups are recovered. Moreover, a weighted L^2-Rellich type inequality with the boundary term is obtained.

Keywords Stratified Lie group · Hardy inequality · Rellich inequality · Uncertainty principle · Caffarelli–Kohn–Nirenberg inequality · Boundary term

Mathematics Subject Classification 35A23 · 35H20

1 Introduction

Let $\mathbb{G}$ be a stratified Lie group (or a homogeneous Carnot group), with dilation structure δ_λ and Jacobian generators $X_1, \ldots, X_N$, so that N is the dimension of the first stratum

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of $\mathbb{G}$. We refer to [10], or to the recent books [4] or [9] for extensive discussions of stratified Lie groups and their properties. Let Q be the homogeneous dimension of $\mathbb{G}$. The sub-Laplacian on $\mathbb{G}$ is given by

$$\mathcal{L} = \sum_{k=1}^{N} X_k^2. \tag{1.1}$$

It was shown by Folland [10] that the sub-Laplacian has a unique fundamental solution ε,

$$\mathcal{L}\varepsilon = \delta,$$

where δ denotes the Dirac distribution with singularity at the neutral element 0 of $\mathbb{G}$. The fundamental solution $\varepsilon(x, y) = \varepsilon(y^{-1}x)$ is homogeneous of degree $-Q + 2$ and can be written in the form

$$\varepsilon(x, y) = [d(y^{-1}x)]^{2-Q}, \tag{1.2}$$

for some homogeneous d which is called the $\mathcal{L}$-gauge. Thus, the $\mathcal{L}$-gauge is a symmetric homogeneous (quasi-) norm on the stratified group $\mathbb{G} = (\mathbb{R}^n, \circ, \delta_\lambda)$, that is,

- $d(x) > 0$ if and only if $x \neq 0$,
- $d(\delta_\lambda(x)) = \lambda d(x)$ for all $\lambda > 0$ and $x \in \mathbb{G}$,
- $d(x^{-1}) = d(x)$ for all $x \in \mathbb{G}$.

We also recall that the standard Lebesque measure dx on $\mathbb{R}^n$ is the Haar measure for $\mathbb{G}$ (see, e.g. [9, Proposition 1.6.6]). The left invariant vector field X_j has an explicit form and satisfies the divergence theorem, see e.g. [9] for the derivation of the exact formula: more precisely, we can write

$$X_k = \frac{\partial}{\partial x'_k} + \sum_{l=2}^{r} \sum_{m=1}^{N_l} a_{k,m}^{(l)}(x', \ldots, x^{(l-1)}) \frac{\partial}{\partial x_m^{(l)}}, \tag{1.3}$$

with $x = (x', x^{(2)}, \ldots, x^{(r)})$, where r is the step of $\mathbb{G}$ and $x^{(l)} = (x_1^{(l)}, \ldots, x_{N_l}^{(l)})$ are the variables in the l^{th} stratum, see also [9, Section 3.1.5] for a general presentation. The horizontal gradient is given by

$$\nabla_{\mathbb{G}} := (X_1, \ldots, X_N),$$

and the horizontal divergence is defined by

$$\mathrm{div}_{\mathbb{G}} v := \nabla_{\mathbb{G}} \cdot v.$$

The horizontal p-sub-Laplacian is defined by

$$\mathcal{L}_p f := \mathrm{div}_{\mathbb{G}}(|\nabla_{\mathbb{G}} f|^{p-2} \nabla_{\mathbb{G}} f), \quad 1 < p < \infty, \tag{1.4}$$

and we will write

$$|x'| = \sqrt{x_1'^2 + \ldots + x_N'^2}$$

for the Euclidean norm on $\mathbb{R}^N$.

Throughout this paper $\Omega \subset \mathbb{G}$ will be an admissible domain, that is, an open set $\Omega \subset \mathbb{G}$ is called an *admissible domain* if it is bounded and if its boundary $\partial\Omega$ is piecewise smooth and simple i.e., it has no self-intersections. The condition for the boundary to be simple amounts to $\partial\Omega$ being orientable.

We now recall the divergence formula in the form of [19, Proposition 3.1]. Let $f_k \in C^1(\Omega) \bigcap C(\overline{\Omega})$, $k = 1, \ldots, N$. Then for each $k = 1, \ldots, N$, we have

$$\int_{\Omega} X_k f_k dz = \int_{\partial\Omega} f_k \langle X_k, dz \rangle. \tag{1.5}$$

Consequently, we also have

$$\int_{\Omega} \sum_{k=1}^{N} X_k f_k dz = \int_{\partial\Omega} \sum_{k=1}^{N} f_k \langle X_k, dz \rangle. \tag{1.6}$$

Using the divergence formula analogues of Green's formulae were obtained in [19] for general Carnot groups and in [20] for more abstract settings (without the group structure), for another formulation see also [11].

The analogue of Green's first formula for the sub-Laplacian was given in [19] in the following form: if $v \in C^1(\Omega) \cap C(\overline{\Omega})$ and $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$, then

$$\int_{\Omega} \left((\widetilde{\nabla} v) u + v \mathcal{L} u \right) dz = \int_{\partial\Omega} v \langle \widetilde{\nabla} u, dz \rangle, \tag{1.7}$$

where

$$\widetilde{\nabla} u = \sum_{k=1}^{N} (X_k u) X_k,$$

and

$$\int_{\partial\Omega} \sum_{k=1}^{N} \langle v X_k u X_k, dz \rangle = \int_{\partial\Omega} v \langle \widetilde{\nabla} u, dz \rangle.$$

Rewriting (1.7) we have

$$\int_{\Omega} \left((\widetilde{\nabla} u) v + u \mathcal{L} v \right) dz = \int_{\partial\Omega} u \langle \widetilde{\nabla} v, dz \rangle,$$
$$\int_{\Omega} \left((\widetilde{\nabla} v) u + v \mathcal{L} u \right) dz = \int_{\partial\Omega} v \langle \widetilde{\nabla} u, dz \rangle.$$

By using $(\widetilde{\nabla} u)v = (\widetilde{\nabla} v)u$ and subtracting one identity for the other we get Green's second formula for the sub-Laplacian:

$$\int_{\Omega} (u\mathcal{L}v - v\mathcal{L}u)dz = \int_{\partial\Omega} (u\langle \widetilde{\nabla} v, dz\rangle - v\langle \widetilde{\nabla} u, dz\rangle). \tag{1.8}$$

It is important to note that the above Green's formulae also hold for the fundamental solution of the sub-Laplacian as in the case of the fundamental solution of the (Euclidean) Laplacian since both have the same behaviour near the singularity $z = 0$ (see [1, Proposition 4.3]).

Weighted Hardy and Rellich inequalities in different related contexts have been recently considered in [15] and [13]. For the general importance of such inequalities we can refer to [2]. Some boundary terms have appeared in [24]. For these inequalities in the setting of general homogeneous groups we refer to [22].

The main aim of this paper is to give the generalised weighted L^p-Hardy and L^p-Rellich type inequalities on stratified groups. In Sect. 2, we present a weighted L^p-Caffarelli–Kohn–Nirenberg type inequality with boundary term on stratified group $\mathbb{G}$, which implies, in particular, the weighted L^p-Hardy type inequality. As consequences of those inequalities, we recover most of the known Hardy type inequalities and Heisenberg–Pauli–Weyl type uncertainty principles on stratified group $\mathbb{G}$ (see [21] for discussions in this direction). In Sect. 3, a weighted L^p-Rellich type inequality is investigated. Moreover, a weighted L^2-Rellich type inequality with the boundary term is obtained together with its consequences.

Usually, unless we state explicitly otherwise, the functions u entering all the inequalities are complex-valued.

2 Weighted L^p-Hardy type inequalities with boundary terms and their consequences

In this section we derive several versions of the L^p weighted Hardy inequalities.

2.1 Weighted L^p-Cafferelli-Kohn-Nirenberg type inequalities with boundary terms

We first present the following weighted L^p-Cafferelli–Kohn–Nirenberg type inequalities with boundary terms on the stratified Lie group $\mathbb{G}$ and then discuss their consequences. The proof of Theorem 2.1 is analogous to the proof of Davies and Hinz [8], but is now carried out in the case of the stratified Lie group $\mathbb{G}$. The boundary terms also give new addition to the Euclidean results in [8]. The classical Caffarelli–Kohn–Nirenberg inequalities in the Euclidean setting were obtained in [6].

Let $\mathbb{G}$ be a stratified group with N being the dimension of the first stratum, and let V be a real-valued function in $L^1_{loc}(\Omega)$ with partial derivatives of order up to 2 in $L^1_{loc}(\Omega)$, and such that $\mathcal{L}V$ is of one sign. Then we have:

Theorem 2.1 *Let Ω be an admissible domain in the stratified group $\mathbb{G}$, and let V be a real-valued function such that $\mathcal{L}V < 0$ holds a.e. in Ω. Then for any complex-valued*

$u \in C^2(\Omega) \cap C^1(\overline{\Omega})$, *and all* $1 < p < \infty$, *we have the inequality*

$$\left\| |\mathcal{L}V|^{\frac{1}{p}} u \right\|^p_{L^p(\Omega)} \leq p \left\| \frac{|\nabla_{\mathbb{G}} V|}{|\mathcal{L}V|^{\frac{p-1}{p}}} |\nabla_{\mathbb{G}} u| \right\|_{L^p(\Omega)} \left\| |\mathcal{L}V|^{\frac{1}{p}} u \right\|^{p-1}_{L^p(\Omega)} - \int_{\partial\Omega} |u|^p \langle \widetilde{\nabla} V, dx \rangle. \tag{2.1}$$

Note that if u vanishes on the boundary $\partial\Omega$, then (2.1) extends the Davies and Hinz result [8] to the weighted L^p-Hardy type inequality on stratified groups:

$$\left\| |\mathcal{L}V|^{\frac{1}{p}} u \right\|_{L^p(\Omega)} \leq p \left\| \frac{|\nabla_{\mathbb{G}} V|}{|\mathcal{L}V|^{\frac{p-1}{p}}} |\nabla_{\mathbb{G}} u| \right\|_{L^p(\Omega)}, \quad 1 < p < \infty. \tag{2.2}$$

Proof of Theorem 2.1 Let $\upsilon_\epsilon := (|u|^2 + \epsilon^2)^{\frac{1}{2}} - \epsilon$. Then $\upsilon_\epsilon^p \in C^2(\Omega) \cap C^1(\overline{\Omega})$ and using Green's first formula (1.7) and the fact that $\mathcal{L}V < 0$ we get

$$\begin{aligned}
\int_\Omega |\mathcal{L}V| \upsilon_\epsilon^p dx &= -\int_\Omega \mathcal{L}V \upsilon_\epsilon^p dx \\
&= \int_\Omega (\widetilde{\nabla} V) \upsilon_\epsilon^p dx - \int_{\partial\Omega} \upsilon_\epsilon^p \langle \widetilde{\nabla} V, dx \rangle \\
&= \int_\Omega \nabla_{\mathbb{G}} V \cdot \nabla_{\mathbb{G}} \upsilon_\epsilon^p dx - \int_{\partial\Omega} \upsilon_\epsilon^p \langle \widetilde{\nabla} V, dx \rangle \\
&\leq \int_\Omega |\nabla_{\mathbb{G}} V| |\nabla_{\mathbb{G}} \upsilon_\epsilon^p| dx - \int_{\partial\Omega} \upsilon_\epsilon^p \langle \widetilde{\nabla} V, dx \rangle \\
&= p \int_\Omega \left(\frac{|\nabla_{\mathbb{G}} V|}{|\mathcal{L}V|^{\frac{p-1}{p}}} \right) |\mathcal{L}V|^{\frac{p-1}{p}} \upsilon_\epsilon^{p-1} |\nabla_{\mathbb{G}} \upsilon_\epsilon| dx - \int_{\partial\Omega} \upsilon_\epsilon^p \langle \widetilde{\nabla} V, dx \rangle,
\end{aligned}$$

where $(\widetilde{\nabla} u) v = \nabla_{\mathbb{G}} u \cdot \nabla_{\mathbb{G}} v$. We have

$$\nabla_{\mathbb{G}} \upsilon_\epsilon = (|u|^2 + \epsilon^2)^{-\frac{1}{2}} |u| \nabla_{\mathbb{G}} |u|,$$

since $0 \leq \upsilon_\epsilon \leq |u|$. Thus,

$$\upsilon_\epsilon^{p-1} |\nabla_{\mathbb{G}} \upsilon_\epsilon| \leq |u|^{p-1} |\nabla_{\mathbb{G}} |u||.$$

On the other hand, let $u(x) = R(x) + iI(x)$, where $R(x)$ and $I(x)$ denote the real and imaginary parts of u. We can restrict to the set where $u \neq 0$. Then we have

$$(\nabla_{\mathbb{G}} |u|)(x) = \frac{1}{|u|} (R(x) \nabla_{\mathbb{G}} R(x) + I(x) \nabla_{\mathbb{G}} I(x)) \quad \text{if} \quad u \neq 0. \tag{2.3}$$

Since

$$\left| \frac{1}{|u|} (R \nabla_{\mathbb{G}} R + I \nabla_{\mathbb{G}} I) \right|^2 \leq |\nabla_{\mathbb{G}} R|^2 + |\nabla_{\mathbb{G}} I|^2, \tag{2.4}$$

we get that $|\nabla_{\mathbb{G}}|u|| \leq |\nabla_{\mathbb{G}} u|$ a.e. in Ω. Therefore,

$$\int_{\Omega} |\mathcal{L}V| \upsilon_{\epsilon}^{p} dx \leq p \int_{\Omega} \left(\frac{|\nabla_{\mathbb{G}} V|}{|\mathcal{L}V|^{\frac{p-1}{p}}} |\nabla_{\mathbb{G}} u| \right) |\mathcal{L}V|^{\frac{p-1}{p}} |u|^{p-1} dx - \int_{\partial\Omega} \upsilon_{\epsilon}^{p} \langle \widetilde{\nabla} V, dx \rangle$$
$$\leq p \left(\int_{\Omega} \left(\frac{|\nabla_{\mathbb{G}} V|^{p}}{|\mathcal{L}V|^{(p-1)}} |\nabla_{\mathbb{G}} u|^{p} \right) dx \right)^{\frac{1}{p}} \left(\int_{\Omega} |\mathcal{L}V| |u|^{p} dx \right)^{\frac{p-1}{p}}$$
$$- \int_{\partial\Omega} \upsilon_{\epsilon}^{p} \langle \widetilde{\nabla} V, dx \rangle,$$

where we have used Hölder's inequality in the last line. Thus, when $\epsilon \to 0$, we obtain (2.1). □

2.2 Consequences of theorem 2.1

As consequences of Theorem 2.1, we can derive the horizontal L^p-Caffarelli–Kohn–Nirenberg type inequality with the boundary term on the stratified group $\mathbb{G}$ which also gives another proof of L^p-Hardy type inequality, and also yet another proof of the Badiale-Tarantello conjecture [3] (for another proof see e.g. [18] and references therein).

2.2.1 Horizontal L^p-Caffarelli–Kohn–Nirenberg inequalities with the boundary term

Corollary 2.2 *Let Ω be an admissible domain in a stratified group $\mathbb{G}$ with $N \geq 3$ being dimension of the first stratum, and let $\alpha, \beta \in \mathbb{R}$. Then for all $u \in C^2(\Omega \backslash \{x' = 0\}) \cap C^1(\overline{\Omega} \backslash \{x' = 0\})$, and any $1 < p < \infty$, we have*

$$\frac{|N-\gamma|}{p} \left\| \frac{u}{|x'|^{\frac{\gamma}{p}}} \right\|_{L^p(\Omega)}^{p} \leq \left\| \frac{\nabla_{\mathbb{G}} u}{|x'|^{\alpha}} \right\|_{L^p(\Omega)} \left\| \frac{u}{|x'|^{\frac{\beta}{p-1}}} \right\|_{L^p(\Omega)}^{p-1}$$
$$- \frac{1}{p} \int_{\partial\Omega} |u|^p \langle \widetilde{\nabla} |x'|^{2-\gamma}, dx \rangle, \tag{2.5}$$

for $2 < \gamma < N$ with $\gamma = \alpha + \beta + 1$, and where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^N$. In particular, if u vanishes on the boundary $\partial\Omega$, we have

$$\frac{|N-\gamma|}{p} \left\| \frac{u}{|x'|^{\frac{\gamma}{p}}} \right\|_{L^p(\Omega)}^{p} \leq \left\| \frac{\nabla_{\mathbb{G}} u}{|x'|^{\alpha}} \right\|_{L^p(\Omega)} \left\| \frac{u}{|x'|^{\frac{\beta}{p-1}}} \right\|_{L^p(\Omega)}^{p-1}. \tag{2.6}$$

Proof of Corollary 2.2 To obtain (2.5) from (2.1) , we take $V = |x'|^{2-\gamma}$. Then

$$|\nabla_{\mathbb{G}} V| = |2-\gamma| |x'|^{1-\gamma}, \qquad |\mathcal{L}V| = |(2-\gamma)(N-\gamma)| |x'|^{-\gamma},$$

and observe that $\mathcal{L}V = (2-\gamma)(N-\gamma)|x'|^{-\gamma} < 0$. To use (2.1) we calculate

$$\left\| |\mathcal{L}V|^{\frac{1}{p}} u \right\|^p_{L^p(\Omega)} = |(2-\gamma)(N-\gamma)| \left\| \frac{u}{|x'|^{\frac{\gamma}{p}}} \right\|^p_{L^p(\Omega)},$$

$$\left\| \frac{|\nabla_{\mathbb{G}} V|}{|\mathcal{L}V|^{\frac{p-1}{p}}} \nabla_{\mathbb{G}} u \right\|_{L^p(\Omega)} = \frac{|2-\gamma|}{|(2-\gamma)(N-\gamma)|^{\frac{p-1}{p}}} \left\| \frac{|\nabla_{\mathbb{G}} u|}{|x'|^{\frac{\gamma-p}{p}}} \right\|_{L^p(\Omega)},$$

$$\left\| |\mathcal{L}V|^{\frac{1}{p}} u \right\|^{p-1}_{L^p(\Omega)} = |(2-\gamma)(N-\gamma)|^{\frac{p-1}{p}} \left\| \frac{u}{|x'|^{\frac{\gamma}{p}}} \right\|^{p-1}_{L^p(\Omega)}.$$

Thus, (2.1) implies

$$\frac{|N-\gamma|}{p} \left\| \frac{u}{|x'|^{\frac{\gamma}{p}}} \right\|^p_{L^p(\Omega)} \leq \left\| \frac{\nabla_{\mathbb{G}} u}{|x'|^{\frac{\gamma-p}{p}}} \right\|_{L^p(\Omega)} \left\| \frac{u}{|x'|^{\frac{\gamma}{p}}} \right\|^{p-1}_{L^p(\Omega)} - \frac{1}{p} \int_{\partial\Omega} |u|^p \langle \widetilde{\nabla} |x'|^{2-\gamma}, dx \rangle.$$

If we denote $\alpha = \frac{\gamma-p}{p}$ and $\frac{\beta}{p-1} = \frac{\gamma}{p}$, we get (2.5). □

2.2.2 Badiale–Tarantello conjecture

Theorem 2.1 also gives a new proof of the generalised Badiale-Tarantello conjecture [3] (see, also [18]) on the optimal constant in Hardy inequalities in $\mathbb{R}^n$ with weights taken with respect to a subspace.

Proposition 2.3 *Let $x = (x', x'') \in \mathbb{R}^N \times \mathbb{R}^{n-N}$, $1 \leq N \leq n$, $2 < \gamma < N$ and $\alpha, \beta \in \mathbb{R}$. Then for any $u \in C_0^\infty(\mathbb{R}^n \backslash \{x' = 0\})$ and all $1 < p < \infty$, we have*

$$\frac{|N-\gamma|}{p} \left\| \frac{u}{|x'|^{\frac{\gamma}{p}}} \right\|^p_{L^p(\mathbb{R}^n)} \leq \left\| \frac{\nabla u}{|x'|^{\alpha}} \right\|_{L^p(\mathbb{R}^n)} \left\| \frac{u}{|x'|^{\frac{\beta}{p-1}}} \right\|^{p-1}_{L^p(\mathbb{R}^n)}, \tag{2.7}$$

where $\gamma = \alpha + \beta + 1$ and $|x'|$ is the Euclidean norm $\mathbb{R}^N$. If $\gamma \neq N$ then the constant $\frac{|N-\gamma|}{p}$ is sharp.

The proof of Proposition 2.3 is similar to Corollary 2.2, so we sketch it only very briefly.

Proof of Proposition 2.3 Let us take $V = |x'|^{2-\gamma}$. We observe that $\Delta V = (2-\gamma)(N-\gamma)|x'|^{-\gamma} < 0$, as well as $|\nabla V| = |2-\gamma||x'|^{(1-\gamma)}$ and $|\Delta V| = |(2-\gamma)(N-\gamma)||x'|^{-\gamma}$. Then (2.1) with

$$\left\| |\Delta V|^{\frac{1}{p}} u \right\|^p_{L^p(\mathbb{R}^n)} = |(2-\gamma)(N-\gamma)| \left\| \frac{u}{|x'|^{\frac{\gamma}{p}}} \right\|^p_{L^p(\mathbb{R}^n)},$$

$$\left\| \frac{|\nabla V|}{|\Delta V|^{\frac{p-1}{p}}} \nabla u \right\|_{L^p(\mathbb{R}^n)} = \frac{|2-\gamma|}{|(2-\gamma)(N-\gamma)|^{\frac{p-1}{p}}} \left\| \frac{\nabla u}{|x'|^{\frac{\gamma-p}{p}}} \right\|_{L^p(\mathbb{R}^n)},$$

$$\left\| |\Delta V|^{\frac{1}{p}} u \right\|_{L^p(\mathbb{R}^n)}^{p-1} = |(2-\gamma)(N-\gamma)|^{\frac{p-1}{p}} \left\| \frac{u}{|x'|^{\frac{\gamma}{p}}} \right\|_{L^p(\mathbb{R}^n)}^{p-1},$$

and denoting $\alpha = \frac{\gamma-p}{p}$ and $\frac{\beta}{p-1} = \frac{\gamma}{p}$, implies (2.7). □

In particular, if we take $\beta = (\alpha+1)(p-1)$ and $\gamma = p(\alpha+1)$, then (2.7) implies

$$\frac{|N-p(\alpha+1)|}{p} \left\| \frac{u}{|x'|^{\alpha+1}} \right\|_{L^p(\mathbb{R}^n)} \leq \left\| \frac{\nabla u}{|x'|^{\alpha}} \right\|_{L^p(\mathbb{R}^n)}, \tag{2.8}$$

where $1 < p < \infty$, for all $u \in C_0^\infty(\mathbb{R}^n \backslash \{x' = 0\})$, $\alpha \in \mathbb{R}$, with sharp constant. When $\alpha = 0$, $1 < p < N$ and $2 \leq N \leq n$, the inequality (2.8) implies that

$$\left\| \frac{u}{|x'|} \right\|_{L^p(\mathbb{R}^n)} \leq \frac{p}{N-p} \|\nabla u\|_{L^p(\mathbb{R}^n)}, \tag{2.9}$$

which given another proof of the Badiale-Tarantello conjecture from [3, Remark 2.3].

2.2.3 The local Hardy type inequality on $\mathbb{G}$.

As another consequence of Theorem 2.1 we obtain the local Hardy type inequality with the boundary term, with d being the $\mathcal{L}$-gauge as in (1.2).

Corollary 2.4 *Let $\Omega \subset \mathbb{G}$ with $0 \notin \partial\Omega$ be an admissible domain in a stratified group $\mathbb{G}$ of homogeneous dimension $Q \geq 3$. Let $0 > \alpha > 2-Q$. Let $u \in C^1(\Omega\backslash\{0\}) \cap C(\overline{\Omega}\backslash\{0\})$. Then we have*

$$\begin{aligned} \frac{|Q+\alpha-2|}{p} \left\| d^{\frac{\alpha-2}{p}} |\nabla_{\mathbb{G}} d|^{\frac{2}{p}} u \right\|_{L^p(\Omega)} \leq & \left\| d^{\frac{p+\alpha-2}{p}} |\nabla_{\mathbb{G}} d|^{\frac{2-p}{p}} |\nabla_{\mathbb{G}} u| \right\|_{L^p(\Omega)} \\ & - \frac{1}{p} \left\| d^{\frac{\alpha-2}{p}} |\nabla_{\mathbb{G}} d|^{\frac{2}{p}} u \right\|_{L^p(\Omega)}^{1-p} \\ & \times \int_{\partial\Omega} d^{\alpha-1} |u|^p \langle \widetilde{\nabla} d, dx \rangle. \end{aligned} \tag{2.10}$$

This extends the local Hardy type inequality that was obtained in [19] for $p = 2$:

$$\begin{aligned} \frac{|Q+\alpha-2|}{2} \left\| d^{\frac{\alpha-2}{2}} |\nabla_{\mathbb{G}} d| u \right\|_{L^2(\Omega)} \leq & \left\| d^{\frac{\alpha}{2}} |\nabla_{\mathbb{G}} u| \right\|_{L^2(\Omega)} \\ & - \frac{1}{2} \left\| d^{\frac{\alpha-2}{2}} |\nabla_{\mathbb{G}} d| u \right\|_{L^2(\Omega)}^{-1} \\ & \times \int_{\partial\Omega} d^{\alpha-1} |u|^2 \langle \widetilde{\nabla} d, dx \rangle. \end{aligned} \tag{2.11}$$

Proof of Corollary 2.4 First, we can multiply both sides of the inequality (2.1) by $\left\| |\mathcal{L}V|^{\frac{1}{p}} u \right\|^{1-p}_{L^p(\Omega)}$, so that we have

$$\left\| |\mathcal{L}V|^{\frac{1}{p}} u \right\|_{L^p(\Omega)} \leq p \left\| \frac{|\nabla_{\mathbb{G}} V|}{|\mathcal{L}V|^{\frac{p-1}{p}}} |\nabla_{\mathbb{G}} u| \right\|_{L^p(\Omega)} - \left\| |\mathcal{L}V|^{\frac{1}{p}} u \right\|^{1-p}_{L^p(\Omega)} \int_{\partial\Omega} |u|^p \langle \widetilde{\nabla} V, dx \rangle. \tag{2.12}$$

Now, let us take $V = d^{\alpha}$. We have

$$\begin{aligned} \mathcal{L}d^{\alpha} = \nabla_{\mathbb{G}}(\nabla_{\mathbb{G}} \varepsilon^{\frac{\alpha}{2-Q}}) = \nabla_{\mathbb{G}} \left(\frac{\alpha}{2-Q} \varepsilon^{\frac{\alpha+Q-2}{2-Q}} \nabla_{\mathbb{G}} \varepsilon \right) \\ = \frac{\alpha(\alpha+Q-2)}{(2-Q)^2} \varepsilon^{\frac{\alpha-4+2Q}{2-Q}} |\nabla_{\mathbb{G}} \varepsilon|^2 + \frac{\alpha}{2-Q} \varepsilon^{\frac{\alpha+Q-2}{2-Q}} \mathcal{L}\varepsilon. \end{aligned}$$

Since ε is the fundamental solution of $\mathcal{L}$, we have

$$\mathcal{L}d^{\alpha} = \frac{\alpha(\alpha+Q-2)}{(2-Q)^2} \varepsilon^{\frac{\alpha-4+2Q}{2-Q}} |\nabla_{\mathbb{G}} \varepsilon|^2 = \alpha(\alpha+Q-2) d^{\alpha-2} |\nabla_{\mathbb{G}} d|^2.$$

We can observe that $\mathcal{L}d^{\alpha} < 0$, and also the identities

$$\begin{aligned} &\left\| |\mathcal{L}d^{\alpha}|^{\frac{1}{p}} u \right\|_{L^p(\Omega)} = \alpha^{\frac{1}{p}} |Q+\alpha-2|^{\frac{1}{p}} \left\| d^{\frac{\alpha-2}{p}} |\nabla_{\mathbb{G}} d|^{\frac{2}{p}} u \right\|_{L^p(\Omega)}, \\ &\left\| \frac{|\nabla_{\mathbb{G}} d^{\alpha}|}{|\mathcal{L}d^{\alpha}|^{\frac{p-1}{p}}} |\nabla_{\mathbb{G}} u| \right\|_{L^p(\Omega)} = \alpha^{\frac{1}{p}} |Q+\alpha-2|^{\frac{1-p}{p}} \left\| d^{\frac{\alpha-2+p}{p}} |\nabla_{\mathbb{G}} d|^{\frac{2-p}{p}} |\nabla_{\mathbb{G}} u| \right\|_{L^p(\Omega)}, \\ &\left\| |\mathcal{L}d^{\alpha}|^{\frac{1}{p}} u \right\|^{1-p}_{L^p(\Omega)} \int_{\partial\Omega} |u|^p \langle \widetilde{\nabla} d^{\alpha}, dx \rangle = \alpha^{\frac{1}{p}} |Q+\alpha-2|^{\frac{1-p}{p}} \left\| d^{\frac{\alpha-2}{p}} |\nabla_{\mathbb{G}} d|^{\frac{2}{p}} u \right\|^{1-p}_{L^p(\Omega)} \\ &\int_{\partial\Omega} d^{\alpha-1} |u|^p \langle \widetilde{\nabla} d, dx \rangle. \end{aligned}$$

Using (2.12) we arrive at

$$\begin{aligned} \frac{|Q+\alpha-2|}{p} \left\| d^{\frac{\alpha-2}{p}} |\nabla_{\mathbb{G}} d|^{\frac{2}{p}} u \right\|_{L^p(\Omega)} \leq \left\| d^{\frac{p+\alpha-2}{p}} |\nabla_{\mathbb{G}} d|^{\frac{2-p}{p}} |\nabla_{\mathbb{G}} u| \right\|_{L^p\Omega} \\ - \frac{1}{p} \left\| d^{\frac{\alpha-2}{p}} |\nabla_{\mathbb{G}} d|^{\frac{2}{p}} u \right\|^{1-p}_{L^p(\Omega)} \int_{\partial\Omega} d^{\alpha-1} |u|^p \langle \widetilde{\nabla} d, dx \rangle, \end{aligned}$$

which implies (2.10). □

2.3 Uncertainty type principles

The inequality (2.12) implies the following Heisenberg-Pauli-Weyl type uncertainty principle on stratified groups.

Corollary 2.5 *Let $\Omega \subset \mathbb{G}$ be admissible domain in a stratified group $\mathbb{G}$ and let $V \in C^2(\Omega)$ be real-valued. Then for any complex-valued function $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ we have*

$$\left\| |\mathcal{L}V|^{-\frac{1}{p}} u \right\|_{L^p(\Omega)} \left\| \frac{|\nabla_{\mathbb{G}} V|}{|\mathcal{L}V|^{\frac{p-1}{p}}} |\nabla_{\mathbb{G}} u| \right\|_{L^p(\Omega)} \geq \frac{1}{p} \|u\|^2_{L^p(\Omega)} + \frac{1}{p} \left\| |\mathcal{L}V|^{-\frac{1}{p}} u \right\|_{L^p(\Omega)} \left\| |\mathcal{L}V|^{\frac{1}{p}} u \right\|^{1-p}_{L^p(\Omega)} \int_{\partial\Omega} |u|^p \langle \widetilde{\nabla} V, dx \rangle. \tag{2.13}$$

In particular, if u vanishes on the boundary $\partial\Omega$, then we have

$$\left\| |\mathcal{L}V|^{-\frac{1}{p}} u \right\|_{L^p(\Omega)} \left\| \frac{|\nabla_{\mathbb{G}} V|}{|\mathcal{L}V|^{\frac{p-1}{p}}} |\nabla_{\mathbb{G}} u| \right\|_{L^p(\Omega)} \geq \frac{1}{p} \|u\|^2_{L^p(\Omega)}. \tag{2.14}$$

Proof of Corollary 2.5 By using the extended Hölder inequality and (2.12) we have

$$\begin{aligned}
&\left\| |\mathcal{L}V|^{-\frac{1}{p}} u \right\|_{L^p(\Omega)} \left\| \frac{|\nabla_{\mathbb{G}} V|}{|\mathcal{L}V|^{\frac{p-1}{p}}} |\nabla_{\mathbb{G}} u| \right\|_{L^p(\Omega)} \\
&\geq \frac{1}{p} \left\| |\mathcal{L}V|^{-\frac{1}{p}} u \right\|_{L^p(\Omega)} \left\| |\mathcal{L}V|^{\frac{1}{p}} u \right\|_{L^p(\Omega)} \\
&\quad + \frac{1}{p} \left\| |\mathcal{L}V|^{-\frac{1}{p}} u \right\|_{L^p(\Omega)} \left\| |\mathcal{L}V|^{\frac{1}{p}} u \right\|^{1-p}_{L^p(\Omega)} \int_{\partial\Omega} |u|^p \langle \widetilde{\nabla} V, dx \rangle, \\
&\geq \frac{1}{p} \left\| |u|^2 \right\|_{L^{\frac{p}{2}}(\Omega)} + \frac{1}{p} \left\| |\mathcal{L}V|^{-\frac{1}{p}} u \right\|_{L^p(\Omega)} \left\| |\mathcal{L}V|^{\frac{1}{p}} u \right\|^{1-p}_{L^p(\Omega)} \int_{\partial\Omega} |u|^p \langle \widetilde{\nabla} V, dx \rangle. \\
&= \frac{1}{p} \|u\|^2_{L^p(\Omega)} + \frac{1}{p} \left\| |\mathcal{L}V|^{-\frac{1}{p}} u \right\|_{L^p(\Omega)} \left\| |\mathcal{L}V|^{\frac{1}{p}} u \right\|^{1-p}_{L^p(\Omega)} \int_{\partial\Omega} |u|^p \langle \widetilde{\nabla} V, dx \rangle,
\end{aligned}$$

proving (2.13). □

By setting $V = |x'|^\alpha$ in the inequality (2.14), we recover the Heisenberg–Pauli–Weyl type uncertainty principle on stratified groups as in [17] and [20]:

$$\left(\int_\Omega |x'|^{2-\alpha} |u|^p dx \right) \left(\int_\Omega |x'|^{\alpha+p-2} |\nabla_{\mathbb{G}} u|^p dx \right) \geq \left(\frac{N+\alpha-2}{p} \right)^p \left(\int_\Omega |u|^p dx \right)^2.$$

In the abelian case $\mathbb{G} = (\mathbb{R}^n, +)$, taking $N = n \geq 3$, for $\alpha = 0$ and $p = 2$ this implies the classical Heisenberg–Pauli–Weyl uncertainty principle for all $u \in C_0^\infty(\mathbb{R}^n \backslash \{0\})$:

$$\left(\int_{\mathbb{R}^n} |x|^2 |u(x)|^2 dx \right) \left(\int_{\mathbb{R}^n} |\nabla u(x)|^2 dx \right) \geq \left(\frac{n-2}{2} \right)^2 \left(\int_{\mathbb{R}^n} |u(x)|^2 dx \right)^2.$$

By setting $V = d^\alpha$ in the inequality (2.14), we obtain another uncertainty type principle:

$$\left(\int_\Omega \frac{|u|^p}{d^{\alpha-2}|\nabla_{\mathbb{G}} d|^2} dx\right)\left(\int_\Omega d^{\alpha+p-2}|\nabla_{\mathbb{G}} d|^{2-p}|\nabla_{\mathbb{G}} u|^p dx\right)$$
$$\geq \left(\frac{Q+\alpha-2}{p}\right)^p \left(\int_\Omega |u|^p dx\right)^2;$$

taking $p = 2$ and $\alpha = 0$ this yields

$$\left(\int_\Omega \frac{d^2}{|\nabla_{\mathbb{G}} d|^2}|u|^2 dx\right)\left(\int_\Omega |\nabla_{\mathbb{G}} u|^2 dx\right) \geq \left(\frac{Q-2}{2}\right)^2 \left(\int_\Omega |u|^2 dx\right)^2.$$

3 Weighted L^p-Rellich type inequalities

In this section we establish weighted Rellich inequalities with boundary terms. We consider first the L^2 and then the L^p cases. The analogous L^2-Rellich inequality on $\mathbb{R}^n$ was proved by Schmincke [23] (and generalised by Bennett [5]).

Theorem 3.1 *Let Ω be an admissible domain in a stratified group $\mathbb{G}$ with $N \geq 2$ being the dimension of the first stratum. If a real-valued function $V \in C^2(\Omega)$ satisfies $\mathcal{L}V(x) < 0$ for all $x \in \Omega$, then for every $\epsilon > 0$ we have*

$$\left\| \frac{|V|}{|\mathcal{L}V|^{\frac{1}{2}}}\mathcal{L}u \right\|^2_{L^2(\Omega)} \geq 2\epsilon \left\| V^{\frac{1}{2}}|\nabla_{\mathbb{G}} u| \right\|^2_{L^2(\Omega)} + \epsilon(1-\epsilon)\left\| |\mathcal{L}V|^{\frac{1}{2}} u \right\|^2_{L^2(\Omega)}$$
$$- \epsilon \int_{\partial\Omega} (|u|^2\langle \widetilde{\nabla} V, dx\rangle - V\langle \widetilde{\nabla}|u|^2, dx\rangle), \tag{3.1}$$

for all complex-valued functions $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$. In particular, if u vanishes on the boundary $\partial\Omega$, we have

$$\left\| \frac{|V|}{|\mathcal{L}V|^{\frac{1}{2}}}\mathcal{L}u \right\|^2_{L^2(\Omega)} \geq 2\epsilon \left\| V^{\frac{1}{2}}|\nabla_{\mathbb{G}} u| \right\|^2_{L^2(\Omega)} + \epsilon(1-\epsilon)\left\| |\mathcal{L}V|^{\frac{1}{2}} u \right\|^2_{L^2(\Omega)}.$$

Proof of Theorem 3.1 Using Green's second identity (1.8) and that $\mathcal{L}V(x) < 0$ in Ω, we obtain

$$\int_\Omega |\mathcal{L}V||u|^2 dx = -\int_\Omega V\mathcal{L}|u|^2 dx - \int_{\partial\Omega}(|u|^2\langle\widetilde{\nabla}V, dx\rangle - V\langle\widetilde{\nabla}|u|^2, dx\rangle)$$
$$= -2\int_\Omega V\left(\mathrm{Re}(\overline{u}\mathcal{L}u) + |\nabla_{\mathbb{G}} u|^2\right) dx$$
$$- \int_{\partial\Omega}(|u|^2\langle\widetilde{\nabla}V, dx\rangle - V\langle\widetilde{\nabla}|u|^2, dx\rangle).$$

Using the Cauchy–Schwartz inequality we get

$$\begin{aligned}\int_\Omega |\mathcal{L}V||u|^2dx &\le 2\left(\frac{1}{\epsilon}\int_\Omega \frac{|V|^2}{|\mathcal{L}V|}|\mathcal{L}u|^2dx\right)^{\frac{1}{2}}\left(\epsilon\int_\Omega |\mathcal{L}V||u|^2dx\right)^{\frac{1}{2}}\\ &\quad - 2\int_\Omega V|\nabla_{\mathbb{G}}u|^2dx - \int_{\partial\Omega}(|u|^2\langle\widetilde{\nabla}V, dx\rangle - V\langle\widetilde{\nabla}|u|^2, dx\rangle)\\ &\le \frac{1}{\epsilon}\int_\Omega \frac{|V|^2}{|\mathcal{L}V|}|\mathcal{L}u|^2dx + \epsilon\int_\Omega |\mathcal{L}V||u|^2dx\\ &\quad - 2\int_\Omega V|\nabla_{\mathbb{G}}u|^2dx - \int_{\partial\Omega}(|u|^2\langle\widetilde{\nabla}V, dx\rangle - V\langle\widetilde{\nabla}|u|^2, dx\rangle),\end{aligned}$$

yielding (3.1). □

Corollary 3.2 *Let $\mathbb{G}$ be a stratified group with N being the dimension of the first stratum. If $\alpha > -2$ and $N > \alpha + 4$ then for all $u \in C_0^\infty(\mathbb{G}\backslash\{x' = 0\})$ we have*

$$\int_{\mathbb{G}\backslash\{x'=0\}} \frac{|\mathcal{L}u|^2}{|x'|^\alpha}dx \ge \frac{(N+\alpha)^2(N-\alpha-4)^2}{16}\int_{\mathbb{G}\backslash\{x'=0\}} \frac{|u|^2}{|x'|^{\alpha+4}}dx. \tag{3.2}$$

Proof of Corollary 3.2 Let us take $V(x) = |x'|^{-(\alpha+2)}$ in Theorem 3.1, which can be applied since $x' = 0$ is not in the support of u. Then we have

$$\nabla_{\mathbb{G}}V = -(\alpha+2)|x'|^{-\alpha-4}x', \qquad \mathcal{L}V = -(\alpha+2)(N-\alpha-4)|x'|^{-(\alpha+4)}.$$

Let us set $C_{N,\alpha} := (\alpha+2)(N-\alpha-4)$. Observing that

$$\mathcal{L}V = -C_{N,\alpha}|x'|^{-(\alpha+4)} < 0,$$

for $|x'| \neq 0$, it follows from (3.1) that

$$\begin{aligned}\int_{\mathbb{G}\backslash\{x'=0\}} \frac{|\mathcal{L}u|^2}{|x'|^\alpha}dx &\ge 2C_{N,\alpha}\epsilon\int_{\mathbb{G}\backslash\{x'=0\}} \frac{|\nabla_{\mathbb{G}}u|^2}{|x'|^{\alpha+2}}dx\\ &\quad + C_{N,\alpha}^2\epsilon(1-\epsilon)\int_{\mathbb{G}\backslash\{x'=0\}} \frac{|u|^2}{|x'|^{\alpha+4}}dx.\end{aligned} \tag{3.3}$$

To obtain (3.2), let us apply the L^p-Hardy type inequality (2.2) by taking $V(x) = |x'|^{\alpha+2}$ for $\alpha \in (-2, N-4)$, so that

$$\int_{\mathbb{G}\backslash\{x'=0\}} \frac{|\nabla_{\mathbb{G}}u|^2}{|x'|^{\alpha+2}}dx \ge \frac{(N-\alpha-4)^2}{4}\int_{\mathbb{G}\backslash\{x'=0\}} \frac{|u|^2}{|x'|^{\alpha+4}}dx,$$

and then choosing $\epsilon = (N+\alpha)/4(\alpha+2)$ for (3.3), which is the choice of ϵ that gives the maximum right-hand side. □

We can now formulate the L^p-version of weighted L^p-Rellich type inequalities.

Theorem 3.3 *Let Ω be an admissible domain in a stratified group $\mathbb{G}$. If $0 < V \in C(\Omega)$, $\mathcal{L}V < 0$, and $\mathcal{L}(V^\sigma) \leq 0$ on Ω for some $\sigma > 1$, then for all $u \in C_0^\infty(\Omega)$ we have*

$$\left\| |\mathcal{L}V|^{\frac{1}{p}} u \right\|_{L^p(\Omega)} \leq \frac{p^2}{(p-1)\sigma + 1} \left\| \frac{V}{|\mathcal{L}V|^{\frac{p-1}{p}}} \mathcal{L}u \right\|_{L^p(\Omega)}, \quad 1 \leq p < \infty. \tag{3.4}$$

Theorem 3.3 will follow by Lemma 3.5, by putting $C = \frac{(p-1)(\sigma-1)}{p}$ in Lemma 3.4.

Lemma 3.4 *Let Ω an admissible domain in a stratified group $\mathbb{G}$. If $V \geq 0$, $\mathcal{L}V < 0$, and there exists a constant $C \geq 0$ such that*

$$C \left\| |\mathcal{L}V|^{\frac{1}{p}} u \right\|^p_{L^p(\Omega)} \leq p(p-1) \left\| V^{\frac{1}{p}} |u|^{\frac{p-2}{p}} |\nabla_{\mathbb{G}} u|^{\frac{2}{p}} \right\|^p_{L^p(\Omega)}, \quad 1 < p < \infty, \tag{3.5}$$

for all $u \in C_0^\infty(\Omega)$, then we have

$$(1 + C) \left\| |\mathcal{L}V|^{\frac{1}{p}} u \right\|_{L^p(\Omega)} \leq p \left\| \frac{V}{|\mathcal{L}V|^{\frac{p-1}{p}}} \mathcal{L}u \right\|_{L^p(\Omega)}, \tag{3.6}$$

for all $u \in C_0^\infty(\Omega)$. If $p = 1$ then the statement holds for $C = 0$.

Proof of Lemma 3.4 We can assume that u is real-valued by using the following identity (see [7, p. 176]):

$$\forall z \in \mathbb{C} : |z|^p = \left(\int_{-\pi}^{\pi} |\cos \vartheta|^p d\vartheta \right)^{-1} \int_{\pi}^{-\pi} |\mathrm{Re}(z) \cos \vartheta + \mathrm{Im}(z) \sin \vartheta|^p d\vartheta,$$

which can be proved by writing $z = r(\cos \phi + i \sin \phi)$ and simplifying.

Let $\epsilon > 0$ and set $u_\epsilon := (|u|^2 + \epsilon^2)^{p/2} - \epsilon^p$. Then $0 \leq u_\epsilon \in C_0^\infty$ and

$$\int_\Omega |\mathcal{L}V| u_\epsilon dx = -\int_\Omega (\mathcal{L}V) u_\epsilon dx = -\int_\Omega V \mathcal{L}u_\epsilon dx,$$

where

$$\begin{aligned}
\mathcal{L}u_\epsilon &= \mathcal{L}\left((|u|^2 + \epsilon^2)^{\frac{p}{2}} - \epsilon^p \right) = \nabla_{\mathbb{G}} \cdot (\nabla_{\mathbb{G}}((|u|^2 + \epsilon^2)^{\frac{p}{2}} - \epsilon^p)) \\
&= \nabla_{\mathbb{G}}(p(|u|^2 + \epsilon^2)^{\frac{p-2}{2}} u \nabla_{\mathbb{G}} u) \\
&= p(p-2)(|u|^2 + \epsilon^2)^{\frac{p-4}{2}} u^2 |\nabla_{\mathbb{G}} u|^2 + p(|u|^2 + \epsilon^2)^{\frac{p-2}{2}} |\nabla_{\mathbb{G}} u|^2 \\
&\quad + p(|u|^2 + \epsilon^2)^{\frac{p-2}{2}} u \mathcal{L}u.
\end{aligned}$$

Then

$$\int_\Omega |\mathcal{L}V|u_\epsilon dx = -\int_\Omega \left(p(p-2)u^2(u^2+\epsilon^2)^{\frac{p-4}{2}} + p(u^2+\epsilon^2)^{\frac{p-2}{2}}\right) V|\nabla_{\mathbb{G}} u|^2 dx$$
$$- p\int_\Omega Vu(u^2+\epsilon^2)^{\frac{p-2}{2}}\mathcal{L}u dx.$$

Hence

$$\int_\Omega |\mathcal{L}V|u_\epsilon + \left(p(p-2)u^2(u^2+\epsilon^2)^{\frac{p-4}{2}} + p(u^2+\epsilon^2)^{\frac{p-2}{2}}\right) V|\nabla_{\mathbb{G}} u|^2 dx$$
$$\leq p\int_\Omega V|u|(u^2+\epsilon^2)^{\frac{p-2}{2}}|\mathcal{L}u| dx.$$

When $\epsilon \to 0$, the integrand on the right is bounded by $V(\max|u|^2+1)^{(p-1)/2}\max|\mathcal{L}u|$ and it is integrable because $u \in C_0^\infty(\Omega)$, and so the integral tends to $\int_\Omega V|u|^{p-1}|\mathcal{L}u|dx$ by the dominated convergence theorem. The integrand on the left is non-negative and tends to $|\mathcal{L}V||u|^p + p(p-1)V|u|^{p-2}|\nabla_{\mathbb{G}}u|^2$ pointwise, only for $u \neq 0$ when $p < 2$, otherwise for any x. It then follows by Fatou's lemma that

$$\left\| |\mathcal{L}V|^{\frac{1}{p}} u \right\|^p_{L^p(\Omega)} + p(p-1)\left\| V^{\frac{1}{p}}|u|^{\frac{p-2}{p}}|\nabla_{\mathbb{G}}u|^{\frac{2}{p}} \right\|^p_{L^p(\Omega)} \leq p\left\| V^{\frac{1}{p}}|u|^{\frac{p-1}{p}}|\mathcal{L}u|^{\frac{1}{p}} \right\|^p_{L^p(\Omega)}.$$

By using (3.5), followed by the Hölder inequality, we obtain

$$(1+C)\left\| |\mathcal{L}V|^{\frac{1}{p}} u \right\|^p_{L^p(\Omega)} \leq p\left\| |\mathcal{L}V|^{(p-1)}V^{\frac{1}{p}}|u|^{\frac{p-1}{p}}|\mathcal{L}V|^{-(p-1)}|\mathcal{L}u|^{\frac{1}{p}} \right\|^p$$
$$\leq p\left\| |\mathcal{L}V|^{\frac{1}{p}} u \right\|^{p-1}_{L^p(\Omega)} \left\| \frac{|V|}{|\mathcal{L}V|^{\frac{p-1}{p}}}\mathcal{L}u \right\|_{L^p(\Omega)}.$$

This implies (3.6). ◻

Lemma 3.5 *Let Ω be an admissible domain in a stratified group $\mathbb{G}$. If $0 < V \in C(\Omega)$, $\mathcal{L}V < 0$, and $\mathcal{L}V^\sigma \leq 0$ on Ω for some $\sigma > 1$, then we have*

$$(\sigma-1)\int_\Omega |\mathcal{L}V||u|^p dx \leq p^2\int_{\{x\in\Omega, u(x)\neq 0\}} V|u|^{p-2}|\nabla_{\mathbb{G}}u|^2 dx < \infty, \quad 1<p<\infty, \tag{3.7}$$

for all $u \in C_0^\infty(\Omega)$.

Proof of Lemma 3.5 We shall use that

$$0 \geq \mathcal{L}(V^\sigma) = \sigma V^{\sigma-2}\left((\sigma-1)|\nabla_{\mathbb{G}}V|^2 + V\mathcal{L}V\right), \tag{3.8}$$

and hence

$$(\sigma-1)|\nabla_{\mathbb{G}}V|^2 \leq V|\mathcal{L}V|.$$

Now we use the inequality (2.2) for $p = 2$ to get

$$(\sigma - 1)\int_\Omega |\mathcal{L}V||u|^2 dx \le 4(\sigma - 1)\int_\Omega \frac{|\nabla_{\mathbb{G}} V|^2}{|\mathcal{L}V|}|\nabla_{\mathbb{G}} u|^2 dx \\ \le 4\int_\Omega V|\nabla_{\mathbb{G}} u|^2 dx = 4\int_{\{x\in\Omega; u(x)\neq 0, |\nabla_{\mathbb{G}} u|\neq 0\}} V|\nabla_{\mathbb{G}} u|^2 dx, \tag{3.9}$$

the last equality valid since $|\{x \in \Omega; u(x) = 0, |\nabla_{\mathbb{G}} u| \neq 0\}| = 0$. This proves Lemma 3.5 for $p = 2$.

For $p \neq 2$, put $v_\epsilon = (u^2 + \epsilon^2)^{p/4} - \epsilon^{p/2}$, and let $\epsilon \to 0$. Since $0 \le v_\epsilon \le |u|^{\frac{p}{2}}$, the left-hand side of (3.9), with u replaced by v_ϵ, tends to $(\sigma - 1)\int_\Omega |\mathcal{L}V||u|^p dx$ by the dominated convergence theorem. If $u \neq 0$, then

$$|\nabla_{\mathbb{G}} v_\epsilon|^2 V = \left|\frac{p}{2}u(u^2+\epsilon^2)^{\frac{p-4}{4}}\nabla_{\mathbb{G}} u\right|^2 V.$$

For $\epsilon \to 0$ we obtain

$$|\nabla_{\mathbb{G}} u|^p V = \frac{p^2}{4}|u|^{p-2}|\nabla_{\mathbb{G}} u|^2 V.$$

It follows as in the proof of Lemma 3.4, by using Fatou's lemma, that the right-hand side of (3.9) tends to

$$p^2\int_{\{x\in\Omega; u(x)\neq 0, |\nabla_{\mathbb{G}} u|\neq 0\}} V|u|^{p-2}|\nabla_{\mathbb{G}} u|^2 dx,$$

and this completes the proof. □

Corollary 3.6 *Let $\mathbb{G}$ be a stratified group with N being the dimension of the first stratum. Then for any $2 < \alpha < N$ and all $u \in C_0^\infty(\mathbb{G}\backslash\{x' = 0\})$ we have the inequality*

$$\int_{\mathbb{G}} \frac{|u|^p}{|x'|^\alpha}dx \le C^p_{(N,p,\alpha)}\int_{\mathbb{G}} \frac{|\mathcal{L}u|^p}{|x'|^{\alpha-2p}}dx, \tag{3.10}$$

where

$$C_{(N,p,\alpha)} = \frac{p^2}{(N-\alpha)\left((p-1)N+\alpha-2p\right)}. \tag{3.11}$$

Proof of Corollary 3.6 Let us choose $V = |x'|^{-(\alpha-2)}$ in Theorem 3.3, so that

$$\mathcal{L}V = -(\alpha-2)(N-\alpha)|x'|^{-\alpha},$$

and we note that when $2 < \alpha < N$, we have $\mathcal{L}V < 0$ for $|x'| \neq 0$. Now it follows from (3.4) that

$$(\alpha-2)^p(N-\alpha)^p\int_{\mathbb{G}} \frac{|u|^p}{|x'|^\alpha}dx \le \frac{p^{2p}}{[(p-1)\sigma+1]^p}\int_{\mathbb{G}} \frac{|\mathcal{L}u|^p}{|x'|^{\alpha-2p}}dx. \tag{3.12}$$

By taking $\sigma = (N-2)/(\alpha-2)$, we arrive at

$$\int_{\mathbb{G}} \frac{|u|^p}{|x'|^\alpha} dx \le \frac{p^{2p}}{(N-\alpha)^p\,((p-1)N+\alpha-2p)^p} \int_{\mathbb{G}} \frac{|\mathcal{L}u|^p}{|x'|^{\alpha-2p}} dx,$$

which proves (3.10)–(3.11). □

Corollary 3.7 *Let $\mathbb{G}$ be a stratified Lie group and let $d = \varepsilon^{\frac{1}{2-Q}}$, where ε is the fundamental solution of the sub-Laplacian $\mathcal{L}$. Assume that $Q \ge 3$, $\alpha < 2$, and $Q+\alpha-4 > 0$. Then for all $u \in C_0^\infty(\mathbb{G}\backslash\{0\})$ we have*

$$\frac{(Q+\alpha-4)^2(Q-\alpha)^2}{16} \int_{\mathbb{G}} d^{\alpha-4} |\nabla_{\mathbb{G}} d|^2 |u|^2 dx \le \int_{\mathbb{G}} \frac{d^\alpha}{|\nabla_{\mathbb{G}} d|^2} |\mathcal{L}u|^2 dx. \tag{3.13}$$

The inequality (3.13) was obtained by Kombe [14], but now we get it as an immediate consequence of Theorem 3.3.

Proof of Corollary 3.7 Let us choose $V = d^{\alpha-2}$ in Theorem 3.3. Then

$$\mathcal{L}V = (\alpha-2)(Q+\alpha-4)d^{\alpha-4}|\nabla_{\mathbb{G}} d|^2.$$

Note that for $Q+\alpha-4 > 0$ and $\alpha < 2$, we have $\mathcal{L}V < 0$ for all $x \ne 0$. If $p = 2$ then from (3.4) it follows that

$$(\alpha-2)^2(Q+\alpha-4)^2 \int_{\mathbb{G}} d^{\alpha-4}|\nabla_{\mathbb{G}} d|^2|u|^2 dx \le \frac{16}{(\sigma+1)^2} \int_{\mathbb{G}} \frac{d^\alpha}{|\nabla_{\mathbb{G}} d|^2} |\mathcal{L}u|^2 dx.$$

By taking $\sigma = (Q-2\alpha+2)/(\alpha-2)$ we get

$$\frac{(Q+\alpha-4)^2(Q-\alpha)^2}{16} \int_{\mathbb{G}} d^{\alpha-4} |\nabla_{\mathbb{G}} d|^2 |u|^2 dx \le \int_{\mathbb{G}} \frac{d^\alpha}{|\nabla_{\mathbb{G}} d|^2} |\mathcal{L}u|^2 dx,$$

proving inequality (3.13). □

Remark 3.8 In the abelian case, when $\mathbb{G} \equiv (\mathbb{R}^n, +)$ with $d = |x|$ being the Euclidean norm, and $\alpha = 0$ in inequality (3.13), we recover the classical Rellich inequality [16].

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